MINIMUM CONVEX COVER OF SPECIAL NONORIENTED GRAPHS

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A vertex set $S$ of a graph $G$ is convex if all vertices of every shortest path between two of its vertices are in $S$. We say that $G$ has a convex $p$-cover if $X(G)$ can be covered by $p$ convex sets. The convex cover number of $G$ is the least $p \geq 2$ for which $G$ has a convex $p$-cover. In particular, the nontrivial convex cover number of $G$ is the least $p \geq 2$ for which $G$ has a convex $p$-cover, where every set contains at least 3 elements. In this paper we determine convex cover number and nontrivial convex cover number of special graphs resulting from some graph operations. We examine graphs resulting from join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.

Keywords: nonoriented graphs, convex covers, convex number, operations, join, cartesian product, lexicographic product, corona.

ACOPERIREA CONVEXĂ MINIMĂ A GRAFURILOR SPECIALE NEORIENTATE

Mulțimea de vârfuri $S$ a grafului $G$ se numește convexă dacă pentru orice două vârfuri $x$ și $y$ din $S$ toate vârfurile ce aparțin tuturor lanțurilor de lungime minimă cu extremitățile $x$ și $y$ se conțin în $S$. Se spune că $G$ conține o $p$-acoperire convexă dacă $X(G)$ poate fi acoperită cu $p$ mulțimi convexe. Numărul acoperirii convexe al grafului $G$ este cel mai mic număr $p \geq 2$ pentru care $G$ conține o $p$-acoperire convexă. În particular, numărul acoperirii convexe netriviale al lui $G$ este cel mai mic număr $p \geq 2$ pentru care $G$ conține o $p$-acoperire convexă în care orice mulțime conține din cel puțin 3 vârfuri. În această lucrare noi determinăm numărul acoperirii convexe și numărul acoperirii convexe netriviale a unor clase speciale de grafuri obținute din următoarele operațiuni pe grafuri: suma, produsul cartezian, produsul lexicografic, corona.

Cuvinte-cheie: grafuri neorientate, acoperiri convexe, numărul acoperirii convexe netriviale, operațiuni, suma grafurilor, produs cartezian, produs lexicografic, corona.

Introduction

In this paper we consider only connected and nonoriented graphs. We denote by $G$ a graph with vertex set $X(G)$ and edge set $U(G)$. An edge joining two vertices $x$ and $y$ in $G$ is denoted by $xy$. The distance between vertices $x$ and $y$ in $G$ is denoted by $d(x,y)$. The diameter of a graph is the length of the shortest path between the most distant nodes.

A set $S \subseteq X$ is a clique if any two vertices of $S$ are adjacent in $G$. The neighborhood of a vertex $x$ of $X(G)$ is the set of all vertices $y$ of $X(G)$ such that $x$ and $y$ are adjacent, and it is denoted by $\Gamma(x)$. A vertex $x$ is called simplicial if $\Gamma(x)$ is a clique. Also, a vertex $x$ is called universal if $\Gamma(x) = X \setminus \{x\}$. Let $S$ be a subset of $X(G)$. We say that $G[S]$ is the subgraph of $G$ induced by $S$.

Now we remind some concepts from [1]. The metric segment $(x,y)$ is the set of all vertices lying on a shortest path between vertices $x$ and $y$ in $G$. A set $S \subseteq X$ is called convex if $(x,y) \subseteq S$ for any two vertices $x,y \in S$. The convex hull of $S \subseteq X$, denoted $d-conv(S)$, is the smallest convex set containing $S$.

A family of sets is called convex cover of $G$ and is denoted by $\mathcal{P}(G)$ if the following conditions hold:

(i) Every set of $\mathcal{P}(G)$ is convex in $G$;
(ii) $X(G) = \bigcup_{S \in \mathcal{P}(G)} S$;
(iii) $S \not\subseteq \bigcup_{C \in \mathcal{P}(G)} C$ for every $S \in \mathcal{P}(G)$.
If $|\mathcal{P}(G)| = p$, then this family is called convex $p$-cover of $G$ and is denoted by $\mathcal{P}_p(G)$ [2].

A convex cover $\mathcal{P}(G)$ of graph $G$ is called nontrivial convex cover if every set $S \in \mathcal{P}(G)$ satisfies the inequalities: $3 \leq |S| \leq |X| - 1$. The minimum number of cliques that cover all the vertices of a graph is known as a clique cover number $\theta(G)$, introduced by Berge [3]. Also, convex cover number $\varphi_c(G)$ was defined as the least $p \geq 2$ for which $G$ has a convex $p$-cover [2]. Similarly to $\varphi_c(G)$, we introduced nontrivial convex cover number $\varphi_{cn}(G)$ [4].

Let us remark that there are graphs for which there are no nontrivial convex covers. For instance, every convex simple graph has no nontrivial convex covers. A graph $G$ is called convex simple if it does not contain nontrivial convex set [5]. Note that if $G$ has a nontrivial convex cover, then $\varphi_c(G) \leq \varphi_{cn}(G)$.

The minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ is the convex $p$-cover of graph $G$ such that $p = \varphi_c(G)$. Similarly, the minimum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}}(G)$ and the minimum clique cover $\mathcal{P}_{\theta}(G)$ of $G$ can be defined.

By $P(G)$ we denote a family of convex sets, where $X(G) = \bigcup_{S \in \mathcal{P}(G)} S$. We denote by $\mathcal{P}(P(G))$ a convex cover of $G$ that consists of sets, which belong to $P(G)$. A nonempty subset $S$ of $X(G)$ is a nonconnecting set in $G$ if for every pair of vertices $x, y \in X(G) \backslash S$ with $d(x, y) = 2$ we have $\Gamma(x) \cap \Gamma(y) \cap S = \emptyset$.

A map $p_G: X(G + H) \rightarrow X(G)$, $p_G((g, h)) = g$, is a projection onto $G$ and $p_H: X(G + H) \rightarrow X(H)$, $p_H((g, h)) = h$, is a projection onto $H$, where $G$ and $H$ are two graphs and symbol $*$ represents one of two operations: cartesian product or lexicographic product.

Convex covers of graphs were studied by many mathematicians. Any latest results on graph convex covers are given in papers [2, 4, 6-8]. Deciding whether a graph $G$ has a convex $p$-cover or a nontrivial convex $p$-cover for a fixed $p \geq 2$ is known to be NP-complete [2, 4]. Besides, convexity was studied in some graph operations [9-11]. Further, there is particular interest in establishing of convex cover number and nontrivial convex cover number for special graphs resulting from graph operations, such as join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.

**Preliminary Results**

First, note that for a given $P(G)$ that has no the set $X(G)$ we can easily obtain $\mathcal{P}(P(G))$ by removing from $P(G)$ all sets contained in the union of other sets of the family $P(G)$. It can easily be checked that Propositions 1, 2 and 3 are true.

**Proposition 1.** Let $G$ be a connected graph of order $n \geq 2$. Then for every vertex $x \in X(G)$ there is a convex set $S \subseteq X(G)$ such that $x \in S$ and $|S| = 2$.

**Proposition 2.** Let $G$ be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_{\varphi_c}(G)$ such that for every set $S \in \mathcal{P}_{\varphi_c}(G)$ the condition $|S| \geq 2$ holds.

**Proposition 3.** Let $G$ be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_{\theta}(G)$ such that for every set $S \in \mathcal{P}_{\theta}(G)$ the condition $|S| \geq 2$ holds.

**Theorem 1.** Let $G$ be a connected graph of order $n \geq 3$ that contains a universal vertex. Then for every vertex $g \in X(G)$ there is a convex set $S \subseteq X(G)$ such that $g \in S$ and $|S| = 3$.

**Proof.** Let $x$ be a universal vertex of $G$, i.e. $\Gamma(x) = X(G) \backslash \{x\}$. Suppose that $G[\Gamma(x)]$ is a disconnected graph. Further, for any two vertices $x_1 \in X(G_1)$ and $x_2 \in X(G_2)$, where $G_1$ and $G_2$ are two different connected components of $G[\Gamma(x)]$, we get a convex set $\{x, x_1, x_2\}$ that is nontrivial.
Now suppose that $G[\Gamma(x)]$ is a connected graph. In this case, every vertex $y$ of $X(G)\setminus\{x\}$ has an adjacent vertex $z \in X(G)\setminus\{x\}$. Hence, set $\{x,y,z\}$ is convex and consists of three vertices. □

**Consequence 1.** Let $G$ be a connected graph of order $n \geq 4$ that contains a universal vertex. Then $G$ has a nontrivial convex cover.

**Consequence 2.** Let $G$ be a connected graph of order $n \geq 4$ that contains a universal vertex. Then $\varphi_c(G) = \varphi_{cn}(G)$.

**Join of Graphs**

The *join* of graphs $G$ and $H$, denoted $G \cup H$, is the graph with vertex set $X(G) \cup X(H)$ and edge set $U(G \cup H) = U(G) \cup U(H) \cup \{xy \mid x \in X(G), y \in X(H)\}$.

**Theorem 2** [9]. Let $G$ be a connected graph and $K_m$ the complete graph of order $m$. Then a proper convex set $C = S_1 \cup S_2$ of $X(G \cup K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$, is convex in $G \cup K_m$ if and only if either

(i) $S_1$ is a clique in $G$, or

(ii) $S_1 \subseteq X(G) \setminus S$ and $S_2 = X(K_m)$ for some nonconnecting set $S$ of $G$.

**Theorem 3.** Let $G$ be a noncomplete graph with diameter 2 and $K_m$ the complete graph of order $m \geq 1$. Let $C = S_1 \cup S_2$ be a proper convex subset of $X(G \cup K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$. Then $S_1$ is convex in $G$.

Proof. By Theorem 2, let us consider two cases. Firstly, if $S_1$ induces a complete subgraph of $G$, then evidently it is convex in $G$. It can be assumed that $S_1$ does not induce a complete subgraph of $G$. Thus, $S_1 \subseteq X(G) \setminus S$ and $S_2 = X(K_m)$ for some nonconnecting set $S$ of $G$. Assume further that $S_1$ is not convex in $G$. Let $x$ and $y$ be two vertices of $S_1$ such that there exists a vertex $z \in \langle x,y \rangle_G$ that does not belong to $S_1$. Since the diameter of $G$ is 2, we obtain $d_G(x,y) = d_{G+K_m}(x,y) = 2$ and $z \in \Gamma_G(x) \cap \Gamma_G(y)$. Hence, $z \in S_1$. From the definition of nonconnecting set, $\Gamma_G(x) \cap \Gamma_G(y) \cap S = \emptyset$ and consequently there is a contradiction. Furthermore, $S_1$ is convex in $G$. □

**Theorem 4.** Let $G$ be a connected graph of order $n \geq 1$ and $K_m$ the complete graph of order $m \geq 1$. Then the following statements hold:

1) If $G$ is complete, then $\varphi_c(G + K_m) = 2$;

2) If $G$ is complete and $n + m \geq 4$, then $\varphi_{cn}(G + K_m) = 2$;

3) If $G$ is noncomplete with diameter 2, then $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m) = \varphi_c(G)$;

4) If $G$ is noncomplete with diameter at least 3, then $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m) \leq \varphi_c(G)$.

Proof.

1) Suppose $G = K_n$. Then, by the definition of the join of two graphs, it follows that $G + K_m$ also is complete. Here graphs $K_n$ and $K_m$ have at least one vertex. Further, we obtain $\varphi_c(K_n + K_m) = 2$.

2) Suppose $G = K_n$ and $n + m \geq 4$. As before, $G + K_m$ is complete. Since every nontrivial convex set has at least three elements, we have $\varphi_{cn}(K_n + K_m) = 2$.

3) Suppose $G$ is a noncomplete graph with diameter 2. Let $C$ be a proper convex subset of $X(G + K_m)$ that satisfies the conditions of Theorem 2. It follows from Theorem 3 that $X(G) \cap C$ is a convex set in $G$. Let $P_{\varphi_c}(G + K_m)$ be a minimum convex cover of $G + K_m$. We obtain a family of sets $P(G) = \{S_i\}$ such that $S_i \subseteq X(G) \setminus C$. Then $\varphi_{cn}(G + K_m) = \varphi_c(G)$.
$\cup_{S \in \mathcal{P}_c(G+K_m)} \{X(G) \cap S\}$. It is clear that $P(G)$ has no the set $X(G)$. This yields that $|\mathcal{P}(P(G))| \leq \varphi_c(G + K_m)$. In fact, we get $\varphi_c(G) \leq \varphi_c(G + K_m)$.

By Proposition 2, a connected graph $G$ on $n \geq 3$ vertices has a minimum convex cover $\mathcal{P}_c(G)$ such that for every set $S \in \mathcal{P}_c(G)$ the condition $|S| \geq 2$ holds. Hence, we obtain a nontrivial convex cover $\mathcal{P}(G + K_m)$ by adding $X(K_m)$ to $Y_i$, where $Y_i \in \mathcal{P}_c(G)$, for every $i$, $1 \leq i \leq \varphi_c(G)$. Note that $|\mathcal{P}(G + K_m)| = \varphi_c(G)$ and $\varphi_c(G + K_m) \leq \varphi_c(G + K_m) \leq \varphi_c(G)$. So, $\varphi_c(G + K_m) = \varphi_c(G + K_m) = \varphi_c(G)$.

4) Now, assume that $G$ is noncomplete and its diameter is at least 3. As above, it is easy to prove that every minimum convex cover of $G$, which satisfies Proposition 2, generates a nontrivial convex cover of $G + K_m$. Hence, $\varphi_{cn}(G + K_m) \leq \varphi_c(G)$. Note also that there are noncomplete graphs $W$, with diameter at least 3, for which the strict inequality $\varphi_{cn}(W + K_m) < \varphi_c(W)$ holds. For instance, graph represented in Figure 1 is the join of graphs $W$ and $K_4$, where $X(K_4) = \{k\}$. This graph has a minimum nontrivial convex cover $\mathcal{P}_c(W + K_4) = \{(x_1, x_7, x_9, k), (x_2, x_6, x_{10}, k), (x_3, x_5, k), (x_4, x_6, k)\}$, but graph $W$ has a minimum convex cover $\mathcal{P}_c(W) = \{(x_1, x_3), (x_5, x_7), (x_2, x_4), (x_6, x_8), (x_9), (x_{10})\}$ and further $\varphi_{cn}(W + K_4) = 4$, but at the same time $\varphi_c(W) = 6$. Hence, $\varphi_{cn}(W + K_4) < \varphi_c(W)$.

We stress that a nontrivial convex cover is a particular case of a convex cover. Since any vertex of $X(K_m)$ is universal in $G + K_m$. Consequence 2 implies $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m)$. Thus, we get $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m) \leq \varphi_c(G)$. □

![Fig.1.](image)

**Theorem 5** [9]. Let $G$ and $H$ be connected noncomplete graphs. Then a proper subset $C = S_1 \cup S_2$ of $X(G + H)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(H)$, is convex in $G + H$ if and only if $S_1$ and $S_2$ are cliques in $G$ and $H$ respectively.

**Theorem 6.** Let $G$ and $H$ be connected noncomplete graphs. Then the following equalities hold:

$$\theta(G + H) = \varphi_c(G + H) = \varphi_{cn}(G + H) = \max\{\theta(G), \theta(H)\}.$$ 

**Proof.** From Theorem 5, we know that every convex set of $G + H$ is a clique. Further, every convex cover of $G + H$ is a clique cover. Therefore, we have $\varphi_c(G + H) = \theta(G + H)$. Let $\mathcal{P}_c(G + H)$ be a minimum convex cover of graph $G + H$. By Theorem 5, we obtain a family of sets $P(G) = \cup_{S \in \mathcal{P}_c(G+H)} \{X(G) \cap S\}$. It is clear that $P(G)$ has no the set $X(G)$ and every set of $P(G)$ is a clique. This implies the inequality $|\mathcal{P}(P(G))| \leq \varphi_c(G + H)$. Thus, $\theta(G) \leq \varphi_c(G + H)$. Continuing in the same way, we see that $|\mathcal{P}(P(H))| \leq \varphi_c(G + H)$ for $P(H) = \cup_{S \in \mathcal{P}_c(G+H)} \{X(H) \cap S\}$, and further $\theta(H) \leq \varphi_c(G + H)$. Hence, we get $\max\{\theta(G), \theta(H)\} \leq \varphi_c(G + H)$. 49
By Proposition 3, we consider minimum clique covers $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ of graphs $G$ and $H$ such that every set of $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ has at least two vertices. If $\theta(G) \geq \theta(H)$ then we construct a nontrivial clique cover $\mathcal{P}(G + H)$ that satisfies the equality $|\mathcal{P}(G + H)| = \theta(G)$. Since every convex set of $G + H$ is a clique, we unify sets $X_i$ and $Y_i$, where $X_i \in \mathcal{P}_\theta(G)$ and $Y_i \in \mathcal{P}_\theta(H)$, for each $i$, $1 \leq i \leq \theta(H)$, and after, unify $X_i$ with $Y_i$ for each $i$, $\theta(H) + 1 \leq i \leq \theta(G)$. By the analogy, if $\theta(G) < \theta(H)$, then it can be constructed a nontrivial clique cover $\mathcal{P}(G + H)$, where $|\mathcal{P}(G + H)| = \theta(H)$. We obtain $\varphi_c(G + H) \leq \varphi_{cn}(G + H) \leq \max\{\theta(G), \theta(H)\}$. So, $\theta(G + H) = \varphi_c(G + H) = \varphi_{cn}(G + H) = \max\{\theta(G), \theta(H)\}$. □

**Cartesian Product of Graphs**

The **cartesian product** of graphs $G$ and $H$ is the graph $G \times H$ on vertex set $X(G) \times X(H)$ in which vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent iff either $g_1 = g_2$ and $h_1 h_2 \in U(H)$ or $h_1 = h_2$ and $g_1 g_2 \in U(G)$.

**Theorem 7** [9]. Let $G$ and $H$ be two connected graphs. A subset $C$ of $X(G) \times X(H)$ is convex in $G \times H$ if and only if $p_G(C)$ is convex in $G$, $p_H(C)$ is convex in $H$, and $C = p_G(C) \times p_H(C)$.

**Theorem 8.** Let $G$ be a connected graph of order $n$ and $K_m$ the complete graph of order $m$ such that $n + m \geq 3$. Then the following statements hold:

1) If $m = 1$, then $\varphi_c(G \times K_m) = \varphi_c(G)$;
2) If $m = 1$ and $n \geq 4$, then $\varphi_{cn}(G \times K_m) = \varphi_{cn}(G)$;
3) If $m \geq 2$, then $\varphi_c(G \times K_m) = 2$;
4) If $m \geq 2$ and $n \geq 3$, or $m \geq 3$ and $n \geq 2$, or $m \geq 4$, then $\varphi_{cn}(G \times K_m) = 2$.

**Proof.**

1) Suppose $m = 1$. Here we see that $G = G \times K_1$. Since $n + m \geq 3$, it is obvious that $\varphi_c(G \times K_m) = \varphi_c(G)$. Further assume that $n \geq 4$. In this case, graph $G \times K_1$ has a nontrivial convex cover if and only if $G$ has a nontrivial convex cover. Consequently, we have $\varphi_{cn}(G \times K_m) = \varphi_{cn}(G)$. So, the statement 2) also holds.

3) Suppose $m \geq 2$. We choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$C_1 = \{(g, k) \mid g \in X(G), k \in X(K_m) \setminus \{k_1\}\}$ and

$C_2 = \{(g, k) \mid g \in X(G), k \in X(K_m) \setminus \{k_2\}\}$.

Since $K_m$ is a complete graph, both sets $C_1$ and $C_2$ satisfy the conditions of Theorem 7. Furthermore, sets $C_1$ and $C_2$ form a convex 2-cover of graph $G \times K_m$ and $\varphi_c(G \times K_m) = 2$. If $n \geq 3$, then we see that $C_1$ and $C_2$ form a nontrivial convex 2-cover of $G \times K_m$ and further $\varphi_{cn}(G \times K_m) = 2$. Similarly, if $m \geq 3$ and $n \geq 2$, or $m \geq 4$, then we also get the equality $\varphi_{cn}(G \times K_m) = 2$. Thus, the statement 4) holds. □

**Theorem 9.** Let $G$ and $H$ be two connected noncomplete graphs and $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \times H)} p_G(S)$. Then $|P(G)| = 1$ or $|P(G) \setminus \{X(G)\}| \geq 2$.

**Proof.** Let $\mathcal{P}_{\varphi_c}(G \times H)$ be a minimum convex cover of $G \times H$. Let $|P(G)| = 1$ and $C \in P(G)$. It means that $C = X(G)$. Now, assume $|P(G) \setminus \{X(G)\}| = 1$. Further, for $S \in P(G) \setminus \{X(G)\}$ there is $S' \in \mathcal{P}_{\varphi_c}(G \times H)$ such that $p_G(S') = S$. If $\{X(G)\} \not\in P(G)$, then we obtain a contradiction, because $X(G), S \neq \emptyset$, which means that $G \times H$ is not covered by convex sets. Suppose further that $\{X(G)\} \in P(G)$. From the definition of convex cover, we know that every set of $\mathcal{P}_{\varphi_c}(G \times H)$ has at least one vertex that belongs only to this set. 50
Hence, there is \( h \in X(H) \) for which there is a vertex \((g,h)\) of \( G \times H \) that belongs to \( S' \) and does not belong to \( S'' \in \mathcal{P}_{\varphi_{c}}(G \times H) \), where \( p_{G}(S'') = X(G) \). By Theorem 7, for \( h \) that we have fixed before and for \( g \in X(G) \setminus S \), vertices \((g,h)\) remain uncovered in \( G \times H \). It is a contradiction. \( \square \)

**Consequence 3.** Let \( G \) and \( H \) be two connected noncomplete graphs and \( P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_{c}}(G \times H)} \{ p_{H}(S) \} \). Then \( |P(H)| = 1 \) or \( |P(H) \setminus \{X(H)\}| \geq 2 \).

**Theorem 10.** Let \( G \) and \( H \) be two connected noncomplete graphs. Then the following equalities hold:

\[
\varphi_{c}(G \times H) = \varphi_{cn}(G \times H) = \min \{ \varphi_{c}(G), \varphi_{c}(H) \}.
\]

**Proof.** First, note that \( |X(G)| \geq 3 \) and \( |X(H)| \geq 3 \). By Proposition 2, there is a minimum convex cover \( \mathcal{P}_{\varphi_{c}}(G) \) of \( G \) such that every set of \( \mathcal{P}_{\varphi_{c}}(G) \) has at least two elements. Further, by Theorem 7, we obtain a nontrivial convex cover \( \mathcal{P}(G \times H) \) that consists of sets \( C_{i} = \{(g,h)\} | g \in S_{i}, h \in X(H) \) \), where \( S_{i} \in \mathcal{P}_{\varphi_{c}}(G) \), \( 1 \leq i \leq \varphi_{c}(G) \). Note that \( |\mathcal{P}(G \times H)| = \varphi_{c}(G) \). This implies \( \varphi_{cn}(G \times H) \leq \varphi_{c}(G) \). For the same reason, if \( \mathcal{P}_{\varphi_{c}}(H) \) is a minimum convex cover of \( H \), then we obtain a nontrivial convex cover \( \mathcal{P}(G \times H) \) of \( G \times H \) such that \( |\mathcal{P}(G \times H)| = \varphi_{c}(H) \) and further \( \varphi_{cn}(G \times H) \leq \varphi_{c}(H) \). Then it stands to reason that \( \varphi_{c}(G \times H) \leq \varphi_{cn}(G \times H) \leq \min \{ \varphi_{c}(G), \varphi_{c}(H) \} \).

Let \( \mathcal{P}_{\varphi_{c}}(G \times H) \) be a minimum convex cover of graph \( G \times H \). Using Theorem 7, we get two families of sets: \( P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_{c}}(G \times H)} \{ p_{G}(S) \} \) and \( P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_{c}}(G \times H)} \{ p_{H}(S) \} \). Evidently, equalities \( |P(G)| = 1 \) and \( |P(H)| = 1 \) do not hold at the same time. By Theorem 9 and Consequence 3, we consider three cases:

Suppose \( |P(G)| = 1 \). In this case, the inequality \( |P(H) \setminus \{X(H)\}| \geq 2 \) holds. Consequently, for a convex cover \( \mathcal{P}(P(H)) \) we get \( |\mathcal{P}(P(H))| \leq \varphi_{c}(G \times H) \) and \( \varphi_{c}(H) \leq \varphi_{c}(G \times H) \). Now, suppose \( |P(H)| = 1 \). As above, we have \( \mathcal{P}(P(G)) \leq \varphi_{c}(G \times H) \) and \( \varphi_{c}(H) \leq \varphi_{c}(G \times H) \). Similarly, if \( |P(G) \setminus \{X(G)\}| \geq 2 \) and \( |P(H) \setminus \{X(H)\}| \geq 2 \), we get \( \varphi_{c}(G) \leq \varphi_{c}(G \times H) \) and \( \varphi_{c}(H) \leq \varphi_{c}(G \times H) \). Combining these three cases, we obtain \( \min \{ \varphi_{c}(G), \varphi_{c}(H) \} \leq \varphi_{c}(G \times H) \). Finally, \( \varphi_{c}(G \times H) = \varphi_{cn}(G \times H) = \min \{ \varphi_{c}(G), \varphi_{c}(H) \} \). \( \square \)

**Lexicographic Product of Graphs**

The lexicographic product of graphs \( G \) and \( H \), denoted \( G \circ H \), is the graph on vertex set \( X(G \circ H) = X(G) \times X(H) \), where vertices \((g_{1}, h_{1})\) and \((g_{2}, h_{2})\) are adjacent if and only if either \( g_{1}g_{2} \in U(G) \) or \( g_{1} = g_{2} \) and \( h_{1}h_{2} \in U(H) \). The graph \( G \circ H \) is called nontrivial if both graphs have at least two vertices.

**Theorem 11** [11]. Let \( C \) be a proper subset of a nontrivial connected lexicographic product \( G \circ H \). If \( C \) induces a noncomplete subgraph of \( G \circ H \), then \( C \) is convex if and only if the following conditions hold:

(i) \( p_{G}(C) \) is convex in \( G \);

(ii) \( \{ g \} \times X(H) \subseteq C \) for every nonsimplicial vertex \( g \in p_{G}(C) \);

(iii) \( H \) is complete.

**Consequence 4.** Let \( C \) be a proper subset of a nontrivial connected lexicographic product \( G \circ H \), where \( H \) is noncomplete. Then \( C \) is convex if and only if it induces a complete subgraph of \( G \circ H \) and the following conditions hold:

(i) \( p_{G}(C) \) induces a complete subgraph of \( G \),

(ii) For every \( g \in p_{G}(C) \), set \( p_{H}(C^{g}) \) induces a complete subgraph of \( H \), where \( C^{g} = \{(g, h) \in C | \text{ for any } h \in H \} \).
Theorem 12. Let $G$ be a connected graph of order $n$ and $K_m$ the complete graph of order $m$ such that $n + m \geq 3$. Then the following statements hold:

1) If $G$ is complete, then $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = 2$;

2) If $G$ is complete and $n + m \geq 5$, or $n = 2$ and $m = 2$, then $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = 2$;

3) If $G$ is noncomplete and $m = 1$, then $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = \varphi_c(G)$;

4) If $G$ is noncomplete, $n \geq 4$ and $m = 1$, then $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_{cn}(G)$;

5) If $G$ is noncomplete, has a simplicial vertex and $m \geq 2$, then $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$;

6) If $G$ is noncomplete, has no simplicial vertices, $m \geq 2$, then $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$;

7) If $G$ is noncomplete and $m \geq 2$, then $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$.

Proof. 

1) Suppose $G$ is complete. Then, it is obvious that the obtained graph is complete and we get equalities $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = 2$. In addition, suppose $n + m \geq 5$, or $n = 2$ and $m = 2$. The obtained complete graph with at least 4 vertices has a nontrivial convex 2-cover. So, $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = 2$. The statement 2) also holds.

3) Suppose $G$ is noncomplete. If $m = 1$, then graphs $G \circ K_m$ and $K_m \circ G$ are equal to $G$ and further we have $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = \varphi_c(G)$. In the same way, with condition $n \geq 4$, the statement 4) holds. In other words, $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_{cn}(G)$. Assume that $m \geq 2$. If $G$ has a simplicial vertex $g'$, then we choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = \{(X(G) \setminus\{g'\}) \times X(K_m)\} \cup \{(g', k) | k \in X(K_m) \setminus\{k_1\}\}$$

$$C_2 = \{(X(G) \setminus\{g'\}) \times X(K_m)\} \cup \{(g', k) | k \in X(K_m) \setminus\{k_2\}\}.$$ 

Evidently, sets $C_1$ and $C_2$ satisfy the conditions of Theorem 11 and these sets form a nontrivial convex 2-cover of $G \circ K_m$. Further, we have $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$. Statement 5) is satisfied.

Now assume that $G$ has no simplicial vertices. We know from Theorem 11 that for every convex set $C$ of $G \circ K_m$ the projection $p_C(G)$ is convex in $G$. Let $\mathcal{P}_{\varphi_c}(G \circ K_m)$ be a minimum convex cover of $G \circ K_m$. We get family $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \circ K_m)} \{p_G(S)\}$. Since noncomplete graph $G$ has no simplicial vertices, it follows that $P(G)$ has no the set $X(G)$. Obviously, for convex cover $\mathcal{P}(P(G))$ of graph $G$ we have $|\mathcal{P}(P(G))| \leq \varphi_c(G \circ H)$. Consequently, $\varphi_c(G) \leq \varphi_c(G \circ K_m)$.

Let $\mathcal{P}_{\varphi_c}(G)$ be a minimum convex cover of $G$. Then, sets $S_i = C_i \times X(K_m)$ form a convex cover of $G \circ K_m$, where $C_i \in \mathcal{P}_{\varphi_c}(G)$. Let $1 \leq i \leq \varphi_c(G)$, and further we get $\varphi_c(G \circ K_m) \leq \varphi_c(G)$. As a result, we have $\varphi_c(G \circ K_m) = \varphi_c(G)$. From Proposition 2 we obtain $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$. So, the statement 6) also holds.

It follows from Consequence 4 that every proper convex subset of $K_m \circ G$ is a clique and further by Proposition 2 and Proposition 3 we have $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$. Furthermore, the statement 7) also holds. □

Theorem 13. Let $G$ and $H$ be two connected noncomplete graphs. Then the following equalities hold:

$$\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta(G \circ H) = \theta(G)\theta(H).$$

Proof. From Consequence 4 we know that every convex set of $G \circ H$ is a clique. Further, we have $\varphi_c(G \circ H) = \theta(G \circ H)$. Moreover, it can be easily checked that $\theta(G \circ H) = \theta(G)\theta(H)$. Taking into account
Proposition 2 and Proposition 3, we get \( \varphi_c(G \circ H) = \varphi_{cn}(G \circ H) \). Finally, \( \varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta(G \circ H) = \theta(G) \theta(H) \). □

**Corona of Graphs**

The *corona* of graphs \( G \) and \( H \) is the graph \( G \circ H \) obtained by taking one copy of \( G \) and \( n \) copies of \( H \), where \( |X(G)| = n \), and then joining by an edge the \( i \)th vertex of \( G \) to every vertex in the \( i \)th copy of \( H \).

We consider a general version of corona of graphs. Let \( G \) be a connected graph on \( n \) vertices. Let \( \{g_1, g_2, \ldots, g_k\} \subseteq X(G) \) and \( H_{g_1}, H_{g_2}, \ldots, H_{g_k}, 1 \leq k \leq n \), be any connected graphs of order at least one. Then by \((G; \{g_1, g_2, \ldots, g_k\}) \circ (H_{g_1}, H_{g_2}, \ldots, H_{g_k}) \) is denoted a graph obtained by taking one copy of \( G \) and, after, by joining every vertex \( g_i \) to every vertex of \( H_{g_i} \), \( 1 \leq i \leq k \). If \( H_{g_1} = H_{g_2} = \cdots = H_{g_k} = H \), then we simply denote \((G; \{g_1, g_2, \ldots, g_k\}) \circ H \). If also \( k = n \), then \((G; \{g_1, g_2, \ldots, g_k\}) \circ H \) is the corona \( G \circ H \).

**Theorem 14** [10]. Let \( G \) be a connected graph and \( H \) be any graph, with \( \{g_1, g_2, \ldots, g_k\} \subseteq X(G) \) and \( H_{g_1}, H_{g_2}, \ldots, H_{g_k} \) being the corresponding copies of \( H \). A nonempty set \( C \subseteq X(G \circ H) \) is convex in \( G \circ H \) if and only if it satisfies one of the following conditions:

(i) \( C \) is a convex set in \( G \);

(ii) \( C \) induces a complete subgraph of \( H_g \) for a vertex \( g \in \{g_1, g_2, \ldots, g_k\} \);

(iii) \((G \circ H)[C] = (G[S]; \{s_1, s_2, \ldots, s_l\}) \circ (H^*_{s_1}, H^*_{s_2}, \ldots, H^*_{s_l}) \), \( S \) is convex in the graph \( G \), \( \{s_1, s_2, \ldots, s_l\} \subseteq S \), \( \{s_1, s_2, \ldots, s_l\} \subseteq \{g_1, g_2, \ldots, g_k\} \) and \( X(s_i + H^*_{s_i}) \) is convex in \( s_i + H^*_{s_i} \) for each \( i \), \( 1 \leq i \leq l \), where \( H^*_{s_i} \) is a subgraph of \( H_{s_i} \).

**Theorem 15.** Let \( G \) and \( H \) be two connected graph of order \( n \) and \( m \), with \( \{g_1, g_2, \ldots, g_k\} \subseteq X(G) \), where \( 1 \leq k \leq n \). Then the following statements hold:

1) If \( n = 1 \), \( H \) is complete, then \( \varphi_c(G \circ H) = 2 \);

2) If \( n = 1 \), \( H \) is complete and \( m \geq 3 \), then \( \varphi_{cn}(G \circ H) = 2 \);

3) If \( n = 1 \), \( H \) is noncomplete with diameter 2, then \( \varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \varphi_c(H) \);

4) If \( n \geq 2 \), \( H \) is noncomplete with diameter at least 3, then \( \varphi_c(G \circ H) = \varphi_{cn}(G \circ H) \leq \varphi_c(H) \);

5) If \( n \geq 2 \), then \( \varphi_c((G; \{g_1, g_2, \ldots, g_k\}) \circ H) = 2 \);

6) If \( n \geq 2 \) and \( k \cdot m + n \geq 4 \), then \( \varphi_{cn}((G; \{g_1, g_2, \ldots, g_k\}) \circ H) = 2 \).

*Proof.* Suppose \( n = 1 \). In fact, \( \varphi_c(K_1 \circ H) = \varphi_c(K_1 + H) \) and the statements 1), 2), 3), 4) hold.

5) Suppose \( n \geq 2 \). It can easily be checked that sets \( X(H_{g_1}) \) and \( X(G) \cup \bigcup_{i=2}^{k} X(H_{g_i}) \) satisfy the conditions of Theorem 14 and further form a convex 2-cover of graph \((G; \{g_1, g_2, \ldots, g_k\}) \circ H \). This implies \( \varphi_c((G; \{g_1, g_2, \ldots, g_k\}) \circ H) = 2 \).

6) Now suppose that \( n \geq 2 \) and \( k \cdot m + n \geq 4 \). In other words, the cardinality of the set \( X((G; \{g_1, g_2, \ldots, g_k\}) \circ H) \) is at least 4. Taking into account Theorem 14, we show the existence of nontrivial convex 2-covers of \((G; \{g_1, g_2, \ldots, g_k\}) \circ H \) in two cases:

a) If \( m = 1 \), then we choose a vertex \( g' \in \Gamma(g) \setminus X(H_g) \) for a vertex \( g \in \{g_1, g_2, \ldots, g_k\} \) and obtain a nontrivial convex 2-cover:

\[
\mathcal{P}_2((G; \{g_1, g_2, \ldots, g_k\}) \circ H) = \{\{g, g'\} \cup X(H_g) \cup \bigcup_{g'' \in \{g_1, g_2, \ldots, g_k\}} X(H_{g''})\}.
\]

b) If \( m \geq 2 \), then we choose a vertex \( h \in H_g \) for a vertex \( g \in \{g_1, g_2, \ldots, g_k\} \) and obtain a nontrivial convex 2-cover:
\[ \mathcal{P}_2((G;\{g_1, g_2, \ldots, g_k\}) \odot H) = \{\{g\} \cup X(H_g), \{h\} \cup X(G) \cup \bigcup_{g' \in \{g_1, g_2, \ldots, g_k\}, g' \neq g} X(H_{g'})\}. \]

The theorem is proved. \( \Box \)

References:


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